JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 26, No. 1, January–February 2003

New Form of Kane's Equations of Motion for Constrained Systems

Abdulrahman H. Bajodah,* Dewey H. Hodges,† and Ye-Hwa Chen‡
Georgia Institute of Technology, Atlanta, Georgia 30332

A differentiated form of the constraint equations, that is, in terms of accelerations, is incorporated into Kane's equations for nonholonomic systems, resulting in equations of motion that are both full order and separated in the generalized accelerations. This means that the time derivatives of all generalized speeds appear in the equations, but only one in each equation. Thus, one obtains a single set of consistent equations without reducing the dimensionality of the space of generalized speeds from the number of generalized coordinates to the number of degrees of freedom. Furthermore, this full dimensionality is maintained without employing Lagrange multipliers. Finally, a systematic method to obtain analytical expressions for the constraint forces is derived. The resulting formulation is believed to be simple and useful in performance analysis and control system design for constrained dynamical systems.

Nomenclature

A, A_1, A_2 = constraint matrices
${}^{\mathcal{R}}\boldsymbol{a}^P$ = acceleration of P in an inertial frame $\mathcal{R}$
B = simple nonholonomic constraint residual matrix
B_i = ith rigid body
C, D = kinematical transformation matrices
$\boldsymbol{F}^c_{P_i}$ = resultant constraint force on the ith particle
$\boldsymbol{F}^c_{Q_j}$ = resultant constraint force on a specified point Q_j of the jth body
F_r, $F^\star_r$ = rth generalized holonomic active, inertia force
F^c_r = rth generalized constraint force
$\tilde{F}_r$, $\tilde{F}^\star_r$ = rth generalized nonholonomic active, inertia force
$\boldsymbol{I}_{B_i}$ = moment of inertia of the ith body parallel to its angular velocity vector
K_q, K_u = kinetic energy gradients
k^{P_i}, k^{B_i}, K = kinetic energy of the ith particle, the ith body, the system
L = acceleration-independent components of holonomic generalized inertia forces column matrix
M = unconstrained inertia matrix
$\boldsymbol{M}^c_{B_j}$ = resultant constraint moment on the jth body
m_{P_i}, m_{B_i} = mass of the ith particle, ith body
N, R = kinetic energy matrices
n = number of generalized coordinates
P_i = ith particle
p = number of degrees of freedom
Q = generalized unconstrained inertia matrix
Q_i = specified point on the ith rigid body
q_r, q = rth generalized coordinate, generalized coordinates column matrix
$\mathcal{R}$ = inertial reference frame
$\boldsymbol{R}_i$ = resultant active force on the ith particle
T, T_w = generalized constrained inertia matrices
$\boldsymbol{T}_i$ = torque on the ith body
t = time
u_I, u_D = independent, dependent generalized speeds column matrix
u_r, u = rth generalized speed, generalized speeds column matrix
$\boldsymbol{v}^{P_i}$, $\boldsymbol{v}^{Q_i}$, $\boldsymbol{v}^{B_i\star}$ = velocity of ith particle, velocity of a specified point Q_i of the ith body, velocity of center of mass of the ith body in an arbitrary reference frame
$\boldsymbol{v}_r$, $\tilde{\boldsymbol{v}}_r$ = rth holonomic, nonholonomic partial velocity
${}^{\mathcal{R}}\boldsymbol{v}^P$ = velocity of P in an inertial frame $\mathcal{R}$
W = inverse of kinematical transformation matrix C
w = transformed generalized speeds column matrix
$\boldsymbol{Z}_i$ = resultant force on a specified point Q_i of the ith body
ν, μ = number of particles, number of bodies comprising the system
Φ_1, Φ_2 = generalized speeds transformation matrices
$\boldsymbol{\omega}_r$, $\tilde{\boldsymbol{\omega}}_r \boldsymbol{\omega}^{B_i}$ = rth holonomic, nonholonomic partial angular velocity, angular velocity of ith body, in an arbitrary reference frame

Introduction

THE first step in dynamic system analysis, design, and/or control is the formulation of a mathematical model. There is no single method to obtain a mathematical model for a dynamic system and no single form of the resulting equations. However, with the continuous increase in complexity of dynamic systems in industry, such as mechatronic systems, chemical plants, and flexible aerospace structures, there is a growing need to enhance the capabilities of modeling techniques for such systems.

In classical mechanics, the modeling process starts by applying the physical laws and constitutive laws, namely, Newton's laws, Lagrange's form of d'Alembert's principle, or Lagrange's equation of motion. Then the process relies on available mathematical tools to cast the equations into the simplest useful form. This renders the modeling process to be dependent on both the application as well as the analyst's ability. The advancement in digital computers has motivated the algorithmic approach of the modeling process. One of the key developments in this arena is the approach popularly referred to as Kane's method and the associated Kane's equations. The major framework was first published in 1965 (see Ref. 1).

Kane's method implements the concept of generalized speeds (quasi-velocity coordinates) as a way to represent motion, similar to what the concept of generalized coordinates does for the configuration. This implementation allows one to focus on the motion aspects of dynamic systems rather than only on the configuration.[2] Therefore, it provides a suitable framework for treating nonholonomic constraints. In the past, such constraints were a hurdle in

Received 19 February 2002; revision received 20 May 2002; accepted for publication 9 September 2002.

*Graduate Student, School of Aerospace Engineering; gt5158a@prism.gatech.edu.
†Professor, School of Aerospace Engineering; dewey.hodges@ae.gatech.edu. Fellow AIAA.
‡Associate Professor, George W. Woodruff School of Mechanical Engineering; yehwa.chen@me.gatech.edu.

the process of obtaining equations of motion for dynamic systems. Generalized speeds provide the formulation process with a desirable flexibility because they can be chosen to satisfy the needs and interests of the designer. The choice of the generalized speeds is crucial because they significantly affect the simplicity of the resulting equations of motion.[3] Historically, it should be recognized that the implicit use of generalized speeds goes back at least to the Gibbs–Appell equations (see Refs. 4 and 5).

The similarities and differences between Gibbs–Appell equations and Kane's equations have been the subjects of interest and debate (see Refs. 6–17). The notion of generalized speeds can be considered the main common feature and the source of power for both. Kane's method adopts a vector approach that inspired useful geometric features of Kane's equations (see Ref. 18). The generalized active forces and generalized inertia forces are obtained by scalar (dot) multiplications of the active and inertia forces, respectively, with the vector entities partial angular velocities and partial velocities. This process delicately eliminates the contribution of constraint forces. We note that Kane's method shares this property with Lagrangean mechanics formulations, but without invoking the principle of virtual work. For the purpose of bringing these forces into evidence, auxiliary generalized speeds, that is, fictitious degrees of freedom that violate the constraints, may be introduced.[2] The resulting equations are simple and effective in describing the motion of nonconservative and nonholonomic systems within the same framework, requiring neither energy methods nor Lagrange multipliers.

Kane's equations have been applied to the formulation of explicit equations of motion for complex flexible structures (Kane and Levinson[19]), as well as to formulate computationally efficient equations of motion in the area of robotics.[20,21] Among the recent applications of Kane's method are the formulations of highly specialized computer-based methodologies for modeling and simulation of multi-rigid- and flexible-body constrained systems[22–27] and the structural dynamic analysis of these systems.[28] The equations are equal in number to the number of degrees of freedom of the system. They can be put in full- or reduced-order form in terms of the dimension of the space of generalized speeds. Making use of known techniques for the nonholonomic, full-order case, one cannot put the equations in a form such that the time derivative of only one generalized speed appears in each equation. In such cases, these equations, together with the kinematical differential equations and the constraint equations, can be solved numerically in two ways. One way is with the aid of differential-algebraic equations (DAE) solvers. This form may cause difficulties for dynamic analysis and control because most of the available time-domain techniques related to these subjects are based on state-space models that are separated in the derivatives of the states (position and velocity variables). In this paper, the acceleration form of constraint equations is utilized to resolve this difficulty. Historically there has been little mention of systematic use of the acceleration form. One reason is perhaps its lack of a "physical" interpretation. The general understanding was that most, if not all, physical constraints are either in the zeroth-order, that is, finite, form or first-order form. The other way, of course, is to rewrite the equations in the reduced-order form and invert the mass matrix to achieve a separated form, state-space model. This is not always desirable for reasons to be specified.

Among the exceptions, Pars[29] adopted the acceleration form (which was called the third form of the fundamental equation) and explored a number of important applications. By suggesting the possibility of a large acceleration change, one is able to analyze problems in which the acceleration is discontinuous (such as a ball rolling off a table). In Ref. 29, the third form was also used to prove Gauss's principle.[30]

More recently, the acceleration form of constraints was utilized in the methods of coordinate partitioning[31,32] and undetermined multipliers[33,34] for constrained dynamic systems. Both methods lead to elimination of the Lagrange multipliers. The resulting equations, together with the acceleration form of constraints, constitute two sets of differential equations. These equations can be integrated simultaneously. They can also be reduced to one set of differential equations in the independent accelerations by using the constraint equations to eliminate the dependent generalized speeds. However, these two sets of differential equations do not constitute, with the kinematic differential equations, a separated-in-accelerations, state-space model description of the dynamic system because more than one acceleration term appears in the same equation. Therefore, this form cannot make use of certain techniques for studying the generic features of dynamic systems, such as stability, chaos, bifurcation, etc. On the other hand, the reduced set of equations can be used to form a state-space model if the constraint equations are used to eliminate the dependent velocities from the kinematic differential equations. This is actually desirable because the resulting equations are free from the constraint drift problem that results from numerical integration. Nevertheless, there are applications under which the appearance of all acceleration terms is important. For example, stability deduced from the reduced form cannot guarantee a stable behavior of the dependent velocity variables. This leaves the designer with the choice of adjoining the unconstrained differential equations with the algebraic constraint equations by a set of Lagrange multipliers and thereby increasing the number of equations and variables. This type of model was used, for example, in the control of nonholonomic systems,[35] the solution of inverse dynamics problems,[36,37] the dynamic analysis of flexible structures,[38,39] and in conjunction with the finite element method for modeling flexible joints in multibody systems.[40] Besides the increase of dimensionality, when using Lagrange multipliers one runs into the difficulties of controlling DAEs and the costly process of solving for and/or controlling the multipliers.

For those reasons, it is highly desirable to obtain a mathematical model that is full order, separated in the accelerations, and involves no Lagrange multipliers. This implies obtaining a nondeficient "constrained" inertia matrix that yields (on inversion) an explicit description of each of the accelerations in terms of only the configuration and kinematic variables and, possibly, time. The first step in this direction was the Udwadia–Kalaba equation.[41,42] The derivation utilizes the acceleration form of the constraint equations, together with the generalized Moore–Penrose inverse of a scaled constraint matrix. Further work in adopting the acceleration form of constraint equations may be found in Refs. 43–46, for example. When the mathematical conformity of the acceleration form of constraints was taken advantage of with the dynamics of the system, explicit expressions for constraint forces were derived without any need to appeal to the free-body approach. This is particularly important in Lagrange's mechanics formulations when the active forces are dependent on the constraint forces, as it is in the case of friction forces.[47]

In the present paper, we take the advantage of the acceleration form of the constraint equations together with the tangential properties of Kane's method to derive a version of Kane's equations for nonholonomic systems that is both full-order and separated in the derivatives of the generalized speeds. The contributions presented are threefold. First, we prove that a square matrix inversion is always possible for all configurations and velocities that satisfy the constraints and all choices of generalized speeds, except for certain configurations of systems that involve toggle positions. Second, Kane's equations can be separated in the time derivative of the generalized speeds in the full-order nonholonomic case. Third, we outline a process for obtaining an explicit form of the constraint forces from these equations without the introduction of auxiliary generalized speeds or Lagrange multipliers.

Unreduced Kane's Equations for Nonholonomic Systems

Consider an inertial reference frame $\mathcal{R}$ in which n generalized coordinates are used to describe the configuration of a set of ν particles and μ rigid bodies forming a nonholonomic system S possessing p degrees of freedom. Let $\boldsymbol{R}_i$ be the resultant active force on the ith particle P_i. The resultant active forces on the ith rigid body B_i are equivalent to a force $\boldsymbol{Z}_i$ on a point Q_i on B_i, together with a torque $\boldsymbol{T}_i$. Kane's equations of motion can be put into the form (Kane and Levinson[2])

$$\tilde{F}_r(q, u, t) + \tilde{F}_r^\star(q, u, \dot{u}, t) = 0, \qquad r = 1, \ldots, p \qquad (1)$$

where q denotes a column matrix containing the generalized coordinates $q_1, \ldots, q_n$ and u denotes a column matrix containing the generalized speeds $u_1, \ldots, u_p$. We refer to $\dot{u}$, the derivative of u with respect to time t, as the generalized acceleration. The generalized speeds u_r satisfy the kinematic differential equations (see Ref. 2, p. 46)

$$\dot{q} = C(q, t)u + D(q, t) \tag{2}$$

where $\dot{q} = \mathrm{d}q/\mathrm{d}t$, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^n$, C^{-1} exists for all $q \in \mathbb{R}^n$, and all $t \in \mathbb{R}$.

Remark: A simple choice of generalized speeds is $u = \dot{q}$, obtained by setting $C(q, t)$ to be the identity matrix $I_{n\times n}$ and $D(q, t)$ a column matrix with n zero elements. For this choice of generalized speeds, the final form of Kane's equations is reduced to Lagrange's equations if equations are formed in full symbolic form. This choice does not usually yield the simplest form of the equations of motion.

To illustrate the method, the generalized speeds are assumed to satisfy the simple nonholonomic/kinematic constraint equations

$$u_{p+r} = \sum_{s=1}^{p} A_{rs}u_s + B_r, \qquad r = 1, \ldots, n - p \tag{3}$$

where the scalars A_{rs} and B_r are functions of the generalized coordinates $q_1, \ldots, q_n$ and t. Let

$$u = \lfloor u_1 \cdots u_n \rfloor^T = \left\lfloor u_I^T \quad u_D^T \right\rfloor^T \tag{4}$$

where $u_I = \lfloor u_1 \cdots u_p \rfloor^T$ and $u_D = \lfloor u_{p+1} \cdots u_n \rfloor^T$. Hence, the matrix representation of Eq. (3) is

$$u_D = A(q, t)u_I + B(q, t) \tag{5}$$

where $A \in \mathbb{R}^{(n-p)\times p}$ and $B \in \mathbb{R}^{n-p}$. The nonholonomic generalized active force $\tilde{F}_r$ is

$$\tilde{F}_r(q, u, t) = \sum_{i=1}^{\nu} (\tilde{F}_r)_{P_i} + \sum_{i=1}^{\mu} (\tilde{F}_r)_{B_i} = \sum_{i=1}^{\nu} \tilde{\boldsymbol{v}}_r^{P_i} \cdot \boldsymbol{R}_i + \sum_{i=1}^{\mu} \tilde{\boldsymbol{v}}_r^{Q_i} \cdot \boldsymbol{Z}_i + \sum_{i=1}^{\mu} \tilde{\boldsymbol{\omega}}_r^{B_i} \cdot \boldsymbol{T}_i, \qquad r = 1, \ldots, p \tag{6}$$

where $\tilde{\boldsymbol{v}}_r^{P_i}$, $\tilde{\boldsymbol{v}}_r^{Q_i}$ are the rth nonholonomic partial velocities of P_i and Q_i, respectively, and $\tilde{\boldsymbol{\omega}}_r^{B_i}$ is the rth nonholonomic partial angular velocity of B_i (Ref. 2). The relation between the holonomic and nonholonomic partial velocities and partial angular velocities are given by (see Ref. 2, P. 48)

$$\tilde{\boldsymbol{v}}_r = \boldsymbol{v}_r + \sum_{s=1}^{n-p} \boldsymbol{v}_{p+s}A_{sr}, \qquad r = 1, \ldots, p \tag{7}$$

and

$$\tilde{\boldsymbol{\omega}}_r = \boldsymbol{\omega}_r + \sum_{s=1}^{n-p} \boldsymbol{\omega}_{p+s}A_{sr}, \qquad r = 1, \ldots, p \tag{8}$$

Therefore, Eq. (6) becomes

$$\tilde{F}_r = \sum_{i=1}^{\nu} \left(\boldsymbol{v}_r^{P_i} + \sum_{s=1}^{n-p} \boldsymbol{v}_{p+s}^{P_i} A_{sr} \right) \cdot \boldsymbol{R}_i + \sum_{i=1}^{\mu} \left(\boldsymbol{v}_r^{Q_i} + \sum_{s=1}^{n-p} \boldsymbol{v}_{p+s}^{Q_i} A_{sr} \right) \cdot \boldsymbol{Z}_i + \sum_{i=1}^{\mu} \left(\boldsymbol{\omega}_r^{Q_i} + \sum_{s=1}^{n-p} \boldsymbol{\omega}_{p+s}^{Q_i} A_{sr} \right) \cdot \boldsymbol{T}_i = F_r + \sum_{s=1}^{n-p} F_{p+s}A_{sr}, \qquad r = 1, \ldots, p \tag{9}$$

In a similar manner, the nonholonomic generalized inertia force is expressed in terms of the holonomic generalized inertia force as

$$\tilde{F}_r^\star = F_r^\star + \sum_{s=1}^{n-p} F_{p+s}^\star A_{sr}, \qquad r = 1, \ldots, p \tag{10}$$

From Eqs. (9) and (10), Eq. (1) can be written as (see Ref. 2, p. 324)

$$F_r + F_r^\star + \sum_{s=1}^{n-p} \left(F_{p+s} + F_{p+s}^\star \right) A_{sr} = 0, \qquad r = 1, \ldots, p \tag{11}$$

Differentiating Eq. (5) with respect to t, and dropping the arguments of the matrices for simplicity, yields

$$\dot{u}_D = \dot{A}u_I + A\dot{u}_I + \dot{B} \tag{12}$$

which can be written as

$$A_1\dot{u} = \dot{A}u_I + \dot{B} \tag{13}$$

where

$$A_1 = [-A \quad I] \tag{14}$$

Equation (11) can be written in matrix form

$$A_2F^\star = -A_2F \tag{15}$$

where

$$A_2 = [I \quad A^T] \tag{16}$$

The velocities and angular velocities of the particles and bodies comprising a dynamic system are linear in the generalized speeds u_r. Hence, the accelerations, angular accelerations, and consequently the generalized inertia forces are linear in $\dot{u}_r$. Therefore, $F^\star$ can be written in the following form:

$$F^\star = -Q(q, t)\dot{u} - L(q, u, t) \tag{17}$$

Then, Eq. (15) becomes

$$A_2Q(q, t)\dot{u} = -A_2L(q, u, t) + A_2F \tag{18}$$

Equations (13) and (18) can be used to form the matrix system

$$T\dot{u} = \left\lfloor [\dot{A}u_I + \dot{B}]^T \quad [A_2(-L + F)]^T \right\rfloor^T \tag{19}$$

where $T = [A_1{}^T \quad [A_2Q]^T]^T$ is invertible, except at toggle positions (where elements of the matrix A reach infinite values). To prove this we note that, because $A_1A_2^T = 0$, the row spaces of A_1 and A_2 are orthogonal complements, with dimensions $n - p$ and p, respectively. It was shown in Ref. 18 that u can be chosen such that Kane's equations are decoupled, that is, each equation contains only one element of $\dot{u}$. This corresponds to the case when A_2Q is the first p rows of the identity matrix $I_{n\times n}$, scaled by the inertia parameters of the system. Because the right $(n - p) \times (n - p)$ submatrix of A_1 is the identity $I_{(n-p)\times(n-p)}$, we conclude that T is invertible for this particular choice of generalized speeds. Denoting these generalized speeds by w and the corresponding matrix T by T_w, we now let u be any set of generalized speeds defined for the system. Then there exists a unique transformation between w and u, as can be seen by equating the right sides of the equations

$$\dot{q} = C_1(q, t)w + D_1(q, t) \tag{20}$$

$$\dot{q} = C_2(q, t)u + D_2(q, t) \tag{21}$$

which gives

$$w = \Phi_1(q, t)u + \Phi_2(q, t) \tag{22}$$

where

$$\Phi_1 = C_1{}^{-1}C_2 \tag{23}$$

$$\Phi_2 = C_1{}^{-1}(D_2 - D_1) \tag{24}$$

Obviously, Φ_1 is invertible. Then we have

$$T\dot{u} = T\Phi_1^{-1}\dot{w} - T\Phi_1^{-1}(\dot{\Phi}_1 u + \dot{\Phi}_2) \tag{25}$$

so that $T_w = T\Phi_1^{-1}$. This transformation implies that the rank of T is no less than that of T_w, which completes the proof for the invertibility of T. Therefore,

$$\dot{u} = T^{-1}\left\lfloor [\dot{A}u_I + \dot{B}]^T \quad [A_2(-L+F)]^T \right\rfloor^T \tag{26}$$

The required inversion of the matrix T in getting from form (19) to form (26) can be done numerically online for the purpose of time simulations. However, to obtain the analytical form of the equations, the symbolic inversion of the matrix T is needed, which cannot be done for excessively large fully populated matrices.

Remark: The holonomic and nonholonomic partial velocities/angular velocities are related to each others by the scalars A_{sr}, as seen from Eqs. (7) and (8). This causes A to appear in the equations of motion (15), besides its appearance in the constraint equations (13). This is a key point in the derivation, because it provides a mathematical relationship between the constraints and the constrained motion.

Remark: The possible configurations of the dynamic system might involve singular configurations that cause numerical explosions of the constraint matrix A for a specific choice of the independent generalized speeds u_I, for example, a toggle position of a four-bar linkage. In this case, different choices of u_I must be employed in the neighborhoods of these configurations such that finite values of the elements of the matrices A_1 and A_2 are obtained and T remains invertible. This is equivalent to the existence of C_I for these configurations. The dependency of the constraint equations does not affect the invertibility of T because A_1 and A_2 always have linearly independent rows, regardless what (finite) values the elements of A might have. Thus, one need not assume that the constraint equations are linearly independent, and any set of constraint equations can be included without the need to extract the largest independent set.

Let us now consider the nature of matrix Q by using the kinetic energy of the system. For a nonholonomic system of ν particles and μ rigid bodies, the rth nonholonomic generalized inertia force can be written as (see Ref. 2, p. 153)

$$\tilde{F}_r^\star = F_r^\star + \sum_{s=1}^{n-p} F_{p+s}^\star A_{sr} = -\sum_{s=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_s}\right) - \frac{\partial K}{\partial q_s}\right] \times\left(W_{sr} + \sum_{k=1}^{n-p} W_{s,p+k}A_{kr}\right), \qquad r = 1,\ldots,p \tag{27}$$

where $W = C^{-1}(q,t)$ and K is the kinetic energy of the system in the reference frame in which the velocities and angular velocities of the system components are defined, given by

$$K = \sum_{i=1}^{\nu} k^{P_i} + \sum_{i=1}^{\mu} k^{B_i} = \frac{1}{2}\sum_{i=1}^{\nu} m_{P_i}(\boldsymbol{v}^{P_i})^2 + \frac{1}{2}\sum_{i=1}^{\mu} m_{B_i}(\boldsymbol{v}^{B_i^\star})^2 + \frac{1}{2}\sum_{i=1}^{\mu}\boldsymbol{I}_{B_i}(\boldsymbol{\omega}^{B_i})^2 \tag{28}$$

Note that K can also be represented as

$$K = \tfrac{1}{2}u^T M(q,t)u + N(q,t)u + R(q,t) \tag{29}$$

where $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{1\times n}$, and $R \in \mathbb{R}$. Hence,

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial K}{\partial u}\frac{\partial u}{\partial \dot{q}} \tag{30}$$

$$\frac{\partial K}{\partial \dot{q}} = [u^T M + N]W \tag{31}$$

Therefore, the matrix representation of Eq. (27) is

$$A_2F^\star = -A_2W^T\left(\frac{\mathrm{d}}{\mathrm{d}t}[W^TMu + W^TN^T] - K_q^T\right) = -A_2W^T\left(W^TM\dot{u} + \frac{\mathrm{d}}{\mathrm{d}t}[W^TM]u + \frac{\mathrm{d}}{\mathrm{d}t}[W^TN^T] - K_q^T\right) \tag{32}$$

where

$$K_q = \lfloor K_{q_1} \ldots K_{q_n} \rfloor \tag{33}$$

$$K_q = \frac{\partial K}{\partial q} \tag{34}$$

If Eq. (17) is multiplied by A_2 and compared with Eq. (32), we obtain

$$Q = W^TW^TM, \tag{35}$$

$$L = W^T\left(\frac{\mathrm{d}}{\mathrm{d}t}[W^TM]u + \frac{\mathrm{d}}{\mathrm{d}t}[W^TN^T] - K_q^T\right) \tag{36}$$

Although Q is not necessarily symmetric, its spectrum set satisfies

$$\mathrm{spec}[Q] = \mathrm{spec}[W^TMW] = \mathrm{spec}[M] \tag{37}$$

Therefore, provided that M is positive definite, Q is of full rank and has positive real eigenvalues. Hence, the row subspaces of A_2 and A_2Q have the same dimension, and the invertibility of T implies that they are the same subspaces.

Remark: Equation (18) can also be obtained by the well-established method of coordinate partitioning.[31] The present method for yielding this result 1) reveals the special structures of matrices A_1 and A_2 and 2) allows us to use the kinematic transformation capability of the generalized speeds to prove the safe invertibility of the augmented coefficients matrix T for suitable choices of u_I and consequently the unreduced form, Eq. (26).

Remark: It is not desirable to use kinetic energy to derive the matrices Q and L. It is easier to obtain $F_r^\star$ directly by constructing the appropriate acceleration and inertia torque vectors.

The procedure of using the acceleration form of constraints in obtaining full-order and separated-in-accelerations equations of motion is summarized as follows:

1) A set of generalized speeds satisfying Eq. (2) is chosen. The dependency among the set is described by Eq. (5). The matrix A is used to construct the matrices A_1 and A_2. If configuration constraints are involved, the corresponding equations are differentiated in time to appear in the same kinematic form.

2) Equation (5) is differentiated in time, resulting in Eq. (13).

3) Expressions are obtained for holonomic partial velocities/angular velocities by inspecting the corresponding expressions for linear/angular velocities, as the coefficients of the generalized speeds.

4) Holonomic generalized active and inertia forces are found from the scalar (dot) product of the impressed and gravitational forces with the holonomic partial velocities/angular velocities, and used together with A_2 to form Eq. (18).

5) Equations (13) and (18) are used to form the matrix equation (19), and T is inverted to yield the resulting equations of motion (26).

Illustrative Examples

The following two examples illustrate the procedure:

Example 1 is that of the motion of a particle subjected to holonomic constraints. Consider the three-dimensional motion of a particle P of mass m as shown in Fig. 1. Let $\mathcal{R}$ be an inertial frame in which the coordinate system (X, Y, Z) is fixed, r be the radial distance from the origin of (X, Y, Z), θ be the polar angle from the X axis, and ϕ be the cone angle from the Z axis. Assume that the particle's motion is restricted by the holonomic constraint $r\phi = c$, where c is a nonzero constant. Because of the nature of the constraint, it is more convenient to use the spherical coordinate system (r, θ, ϕ). Let the particle be at the origin of the coordinate system (X', Y', Z') where X' is along r, attached to a unit vector $\boldsymbol{i}$, and pointing

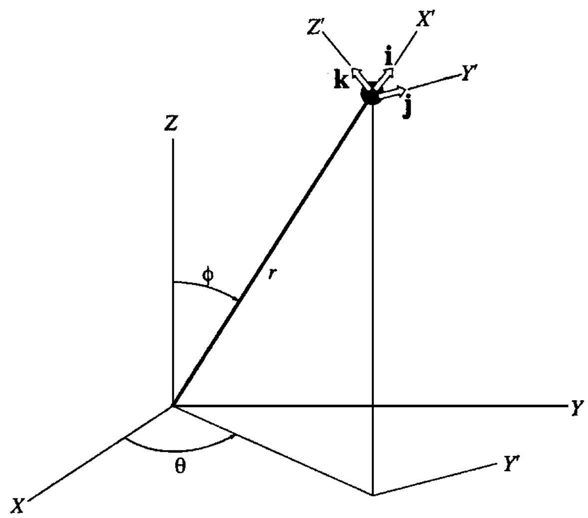

Fig. 1 Schematic for example 1.

outward relative to O; Y' is parallel to the XY plane, attached to a unit vector $\boldsymbol{j}$ pointing to the positive rotation direction relative to the Z axis; and Z' is perpendicular to both x' and y', attached to a unit vector $\boldsymbol{k}$ such that $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$. Assume that the resultant active force on the particle is given by

$$\boldsymbol{F} = F_r \boldsymbol{i} + F_\theta \boldsymbol{j} + F_\phi \boldsymbol{k} \tag{38}$$

The particle's velocity in $\mathcal{R}$ is

$${}^{\mathcal{R}}\boldsymbol{v}^P = \dot{r}\boldsymbol{i} + r\sin\phi\dot{\theta}\boldsymbol{j} - r\dot{\phi}\boldsymbol{k} \tag{39}$$

Let the generalized speeds be $u_1 = \dot{r}$, $u_2 = r\dot{\theta}\sin\phi$, and $u_3 = -r\dot{\phi}$. Then one can write

$${}^{\mathcal{R}}\boldsymbol{v}^P = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k} \tag{40}$$

so that the partial velocities are $\boldsymbol{v}_1 = \boldsymbol{i}$, $\boldsymbol{v}_2 = \boldsymbol{j}$, and $\boldsymbol{v}_3 = \boldsymbol{k}$. The acceleration of the particle in $\mathcal{R}$ is

$${}^{\mathcal{R}}\boldsymbol{a}^P = \frac{{}^{\mathcal{R}}\mathrm{d}{}^{\mathcal{R}}\boldsymbol{v}^P}{\mathrm{d}t} = \left(\dot{u}_1 - \frac{u_2^2 + u_3^2}{r}\right)\boldsymbol{i} + \left(\dot{u}_2 + \frac{u_1u_2}{r} - \frac{u_2u_3}{r\tan\phi}\right)\boldsymbol{j} + \left(\dot{u}_3 + \frac{u_1u_3}{r} + \frac{{u_2}^2}{r\tan\phi}\right)\boldsymbol{k} \tag{41}$$

The generalized inertia forces are

$$F_1^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_1 = -m\left\{\dot{u}_1 - \left[\left(u_2^2 + u_3^2\right)/r\right]\right\} \tag{42}$$

$$F_2^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_2 = -m(\dot{u}_2 + u_1u_2/r - u_2u_3/r\tan\phi) \tag{43}$$

$$F_3^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_3 = -m\left(\dot{u}_3 + u_1u_3/r + u_2^2\big/r\tan\phi\right) \tag{44}$$

In matrix form,

$$F^\star = \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{Bmatrix} m\left[\left(u_2^2 + u_3^2\right)/r\right] \\ m(u_2u_3/r\tan\phi - u_1u_2/r) \\ m\left[-(u_1u_3/r) - u_2^2\big/r\tan\phi\right] \end{Bmatrix} \tag{45}$$

The generalized active forces are

$$F_1 = \boldsymbol{F} \cdot \boldsymbol{v}_1 = F_r \tag{46}$$

$$F_2 = \boldsymbol{F} \cdot \boldsymbol{v}_2 = F_\theta \tag{47}$$

$$F_3 = \boldsymbol{F} \cdot \boldsymbol{v}_3 = F_\phi \tag{48}$$

Differentiating the holonomic constraint $r\phi = c$ with respect to t yields

$$\dot{r}\phi + r\dot{\phi} = 0 \tag{49}$$

or

$$u_3 = \phi u_1 \tag{50}$$

Therefore,

$$A = \lfloor \phi \quad 0 \rfloor \tag{51}$$

$$B = 0 \tag{52}$$

Differentiating the constraint matrix A with respect to t yields

$$\dot{A} = \lfloor -(u_3/r) \quad 0 \rfloor \tag{53}$$

Also,

$$A_1 = [-A \quad I] = \lfloor -\phi \quad 0 \quad 1 \rfloor \tag{54}$$

Thus, Eq. (13) for this system is

$$\lfloor -\phi \quad 0 \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \left\lfloor -\frac{u_3}{r} \quad 0 \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{55}$$

On the other hand,

$$A_2 = [I \quad A^T] \tag{56}$$

$$A_2 = \begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix} \tag{57}$$

so that Eq. (18) for this system is

$$\begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + \begin{Bmatrix} m\left[\left(u_2^2 + u_3^2\right)/r\right] \\ m(u_2u_3/r\tan\phi - u_1u_2/r) \\ m\left[-(u_1u_3/r) - u_2^2\big/r\tan\phi\right] \end{Bmatrix} \right\} = -\begin{bmatrix} 1 & 0 & \phi \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_r \\ F_\theta \\ F_\phi \end{Bmatrix} \tag{58}$$

or

$$\begin{bmatrix} -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} -m\left[\left(u_2^2 + u_3^2\right)/r\right] + m\phi\left(u_1u_3/r + u_2^2\big/r\tan\phi\right) - F_r - \phi F_\phi \\ -m(u_2u_3/r\tan\phi - u_1u_2/r) - F_\theta \end{Bmatrix} \tag{59}$$

Therefore, Eqs. (55) and (59) can be used to form the matrix system

$$\begin{bmatrix} -\phi & 0 & 1 \\ -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} -(u_1u_3/r) \\ -(m/r)\left[u_2^2 + u_3^2 - \phi u_1u_3 - \phi\left(u_2^2\big/\tan\phi\right)\right] - F_r - \phi F_\phi \\ (m/r)(u_1u_2 - u_2u_3/\tan\phi) - F_\theta \end{Bmatrix} \tag{60}$$

When $\dot{u}$ is solved for,

$$\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{bmatrix} -\phi & 0 & 1 \\ -m & 0 & -m\phi \\ 0 & -m & 0 \end{bmatrix}^{-1}$$

$$\begin{Bmatrix} -(u_1u_3/r) \\ -(m/r)\left[u_2^2 + u_3^2 - \phi u_1 u_3 - \phi\left(u_2^2/\tan\phi\right)\right] - F_r - \phi F_\phi \\ (m/r)(u_1u_2 - u_2u_3/\tan\phi) - F_\theta \end{Bmatrix}$$

$$= \frac{1}{a}\begin{bmatrix} -m^2\phi & -m & 0 \\ 0 & 0 & -m(\phi^2+1) \\ m^2 & -m\phi & 0 \end{bmatrix}$$

$$\begin{Bmatrix} -(u_1u_3/r) \\ -(m/r)\left[u_2^2 + u_3^2 - \phi u_1 u_3 - \phi\left(u_2^2/\tan\phi\right)\right] - F_r - \phi F_\phi \\ (m/r)(u_1u_2 - u_2u_3/\tan\phi) - F_\theta \end{Bmatrix} \tag{61}$$

where $a = m^2(1+\phi^2)$. Therefore, the final form of the equations is

$$\dot{u}_1 = (m/a)(F_r + \phi F_\phi) + (m^2/ar)\left[u_2^2 + u_3^2 - \phi\left(u_2^2/\tan\phi\right)\right] \tag{62}$$

$$\dot{u}_2 = (F_\theta/m) - (1/r)(u_1u_2 - u_2u_3/\tan\phi) \tag{63}$$

$$\dot{u}_3 = (m\phi/a)(F_r + \phi F_\phi) + (m^2\phi/ar) \times \left[u_2^2 + u_3^2 - \phi\left(u_2^2/\tan\phi\right)\right] - (u_1u_3/r) \tag{64}$$

Remark: The particle is subjected to one holonomic constraint. Thus two generalized coordinates and two generalized speeds are sufficient to form the equations. However, because the holonomic constraint equation is differentiated with respect to time, the constraint is treated as nonholonomic, and one pseudogeneralized coordinate is added, together with one dependent generalized speed that satisfies Eq. (7).

Example 2 is that of the motion of a particle subjected to nonholonomic constraints. Consider the three-dimensional motion of a particle P of mass m subjected to the nonholonomic constraint

$$\dot{y} = z\dot{x} + \alpha(t) \tag{65}$$

where $\alpha(t)$ is a prescribed smooth function of time. The active forces in the Cartesian coordinate system (x, y, z) are F_x, F_y, and F_z, respectively. Assume that (x, y, z) is fixed to an inertial frame $\mathcal{R}$, and define the generalized speeds as $u_1 = \dot{x}$, $u_2 = \dot{z}$, and $u_3 = \dot{y}$. Then, the preceding nonholonomic constraint equation can be written as

$$u_3 = \lfloor z \quad 0 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \alpha(t) \tag{66}$$

Therefore,

$$A = \lfloor z \quad 0 \rfloor, \qquad B = \alpha(t) \tag{67}$$

Differentiating the constraint matrix A with respect to time yields

$$\dot{A} = \lfloor u_2 \quad 0 \rfloor \tag{68}$$

The matrix A_1 is, therefore,

$$A_1 = [-A \quad I] = \lfloor -z \quad 0 \quad 1 \rfloor \tag{69}$$

Thus, Eq. (13) for this system is

$$\lfloor -z \quad 0 \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \lfloor u_2 \quad 0 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \dot{\alpha} \tag{70}$$

The matrix A_2 is

$$A_2 = [I \quad A^T] = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \tag{71}$$

The velocity of the particle in $\mathcal{R}$ is

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1\boldsymbol{v}_1 + u_2\boldsymbol{v}_2 + u_3\boldsymbol{v}_3 \tag{72}$$

$$^{\mathcal{R}}\boldsymbol{v}^P = \dot{x}\boldsymbol{i} + \dot{z}\boldsymbol{k} + \dot{y}\boldsymbol{j} \tag{73}$$

so that $\boldsymbol{v}_1 = \boldsymbol{i}$, $\boldsymbol{v}_2 = \boldsymbol{k}$, and $\boldsymbol{v}_3 = \boldsymbol{j}$, where $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors parallel to the x, y, and z axes, respectively.

The generalized active forces are $F_1 = F_x$, $F_2 = F_z$, and $F_3 = F_y$. The generalized inertia forces are

$$F_1^\star = -m\dot{u}_1 \tag{74}$$

$$F_2^\star = -m\dot{u}_2 \tag{75}$$

$$F_3^\star = -m\dot{u}_3 \tag{76}$$

In matrix form

$$F^\star = \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} \tag{77}$$

Hence, Eq. (18) for this system is

$$\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -m & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = -\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \\ F_y \end{Bmatrix} \tag{78}$$

or

$$\begin{bmatrix} -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = -\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \\ F_y \end{Bmatrix} \tag{79}$$

Equations (70) and (79) can be put in the following matrix form:

$$\begin{bmatrix} -z & 0 & 1 \\ -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{Bmatrix} u_1u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix} \tag{80}$$

When $\dot{u}$ is solved for,

$$\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{bmatrix} -z & 0 & 1 \\ -m & 0 & -mz \\ 0 & -m & 0 \end{bmatrix}^{-1} \begin{Bmatrix} u_1u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix} \tag{81}$$

$$\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} = \begin{bmatrix} -(mz/a) & -1/a & 0 \\ 0 & 0 & -1/m \\ m/a & -z/a & 0 \end{bmatrix} \begin{Bmatrix} u_1u_2 + \dot{\alpha} \\ -F_x - F_y z \\ -F_z \end{Bmatrix} \tag{82}$$

where $a = m(1+z^2)$. Therefore, the final form of the equations of motion is

$$\dot{u}_1 = (-mz/a)(u_1u_2 + \dot{\alpha}) + (1/a)(F_x + F_y z) \tag{83}$$

$$\dot{u}_2 = F_z/m \tag{84}$$

$$\dot{u}_3 = (m/a)(u_1u_2 + \dot{\alpha}) + (z/a)(F_x + F_y z) \tag{85}$$

Constraint Forces

Similar to Lagrange's fundamental equations, Kane's equations do not identify the constraint forces directly. The way introduced in Ref. 2 to bring these forces into evidence is to define a set of generalized speeds that violate the constraints, without considering more generalized coordinates. This results in an increase in the number of partial velocities and the number of the governing equations, from which the constraint forces and moments are determined. The choice of these generalized speeds is not unique.

In Ref. 33, the same issue is solved by introducing Lagrange-multiplier-like scalars to adjoin the constraint matrix A_1 with Kane's equation for holonomic systems. The resulting equations together with the constraint equations are solved for these scalars and the generalized speeds and, hence, for the constraint forces and moments. This approach is suitable for small systems with few degrees of freedom and small numbers of constraints. If the resulting sets of equations are large and complicated, the solutions for such equations become more difficult. In addition, it is not possible in general to obtain closed-form, analytical solutions for the set of multipliers and, consequently, for the constraint forces, in terms of the generalized coordinates and the generalized speeds.

Lesser[18] suggested the method of projecting the given and inertia forces and moments on the orthogonal complement space of the "configuration space" spanned by the partial angular velocities and the partial velocities. This requires finding a spanning set of vectors to this defined space and solving the resulting complementary equations for the constraint forces and moments. These equations are coupled with Kane's equations in case the given forces are dependent on the constraint forces.

The acceleration form of the constraint equations can be used to obtain simply and systematically explicit analytical expressions for the constraint forces and moments. Kane's equation for holonomic systems is (see Ref. 2, p. 158)

$$F(q,u,t) + F^\star(q,u,\dot{u},t) = 0 \tag{86}$$

Because the constraint equations are differentiated with respect to time in our framework, it can be assumed, without loss of generality, that the dynamic system is subjected to nonholonomic constraints only; hence, in this context, "holonomic" and "unconstrained" have the same meaning. Equation (86) is considered the equation for constraint-free systems in this sense.

Consider now a constrained system S consisting of a set of ν particles and μ rigid bodies. Let the resultant forces and moments exerted on S by the constraints be the force $\boldsymbol{F}^c_{P_i}$ on the ith particle, $\boldsymbol{F}^c_{Q_j}$ on the point Q_j of the jth body, and the moment $\boldsymbol{M}^c_{B_j}$ on the jth body, $i = 1 \cdots \nu$, $j = 1 \cdots \mu$. Also, let $\boldsymbol{v}^{P_i}_r$, $\boldsymbol{v}^{Q_j}_r$, and $\boldsymbol{\omega}^{B_j}_r$ be the rth partial velocity of the ith particle, the rth partial velocity of the point Q_j, and the rth partial angular velocity of the jth body, respectively. These holonomic (unconstrained) partial velocities and partial angular velocities satisfy the relations (see Ref. 2, p. 45)

$$\boldsymbol{v}^{P_i} = \sum_{r=1}^{n} u_r \boldsymbol{v}^{P_i}_r + \boldsymbol{v}^{P_i}_t \tag{87}$$

$$\boldsymbol{v}^{Q_j} = \sum_{r=1}^{n} u_r \boldsymbol{v}^{Q_i}_r + \boldsymbol{v}^{Q_j}_t \tag{88}$$

$$\boldsymbol{\omega}^{B_j} = \sum_{r=1}^{n} u_r \boldsymbol{\omega}^{B_j}_r + \boldsymbol{\omega}^{B_j}_t \tag{89}$$

where n is the number of generalized coordinates, u_rs are the generalized speeds defined for the system, and $\boldsymbol{v}^{P_i}_t$, $\boldsymbol{v}^{Q_j}_t$, and $\boldsymbol{\omega}^{B_j}_t$ are possibly time-varying functions of the generalized coordinates but are independent of the generalized speeds. We define F^c as the generalized constraint force the rth component of which is given by

$$F^c_r = \sum_{i=1}^{\nu} \boldsymbol{F}^c{}_{P_i} \cdot \boldsymbol{v}^{P_i}_r + \sum_{j=1}^{\mu} \boldsymbol{F}^c_{Q_j} \cdot \boldsymbol{v}^{Q_i}_r + \sum_{j=1}^{\mu} \boldsymbol{M}^c_{B_j} \cdot \boldsymbol{\omega}^{B_j}_r \qquad r = 1, \ldots, n \tag{90}$$

Because the deviation in $\dot{u}$ from an unconstrained value can occur only because of the constraint force, an equation for constrained motion can be written by adding F^c to the left-hand side of Eq. (86), so that

$$F(q,u,t) + F^\star(q,u,\dot{u},t) + F^c(q,u,t) = 0 \tag{91}$$

The generalized inertia force column matrix for this system can be written in terms of the kinetic energy as (see Ref. 2, p. 153)

$$F^\star = -W^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} \right]^T \tag{92}$$

If the kinetic energy is represented in the form of Eq. (29), then Eq. (92) becomes

$$F^\star = -W^T \left(W^T M \dot{u} + \frac{\mathrm{d}}{\mathrm{d}t}[W^T M]u + \frac{\mathrm{d}}{\mathrm{d}t}[W^T N^T] - K_q{}^T \right) \tag{93}$$

$$F^\star = -Q\dot{u} - L(q,u,t) \tag{94}$$

where Q and L have the same definitions as in the preceding section. When substituted into Eq. (91),

$$F(q,u,t) - Q\dot{u} - L(q,u,t) + F^c(q,u,t) = 0 \tag{95}$$

Substituting the expression (26) for $\dot{u}$ into the constraints-relaxed equation (95) and solving for F^c gives

$$F^c = -F(q,u,t) + QT^{-1} \times \left\lfloor [\dot{A}u_I + \dot{B}]^T \quad [A_2(-L+F)]^T \right\rfloor^T + L(q,u,t) \tag{96}$$

Although similar results can be obtained by other methods, the present approach does not require the introduction of Lagrange multipliers[31] or of fictitious degrees of freedom that violate the constraints.[2]

Usually, it is not possible to construct the actual constraint forces if more than one of these contribute to F^c. For that reason, it is desirable in forming Eq. (91) to include only the generalized speeds that describe the motion of one constrained point or the angular motion of one constrained body at a time, in addition to including the independent generalized speeds.

In some applications, the generalized constraint forces F^c are those of interest, and it might not be important to calculate the actual constraint load component magnitudes. An example is the subject of servoconstraints, where the constraint equations (5) are requirements to be satisfied and F^c are dependent on control variables that are determined to impose these requirements.

Remark: Generalized constraint forces derived here are inertially based because the velocities and accelerations used to derive Eq. (96) are in an inertial frame. In general, active forces may contain acceleration terms.[48] In such cases, the generalized constraint forces would be dependent on $\dot{u}$. For simplicity, we consider the active forces to be dependent only on q, u, and t, and independent of $\dot{u}$.

The procedure for finding constraint forces is summarized:

1) Generalized active and inertia forces satisfying (86) are derived for the corresponding unconstrained system.

2) Expressions for generalized accelerations are obtained by the procedure outlined in the preceding section.

3) Generalized accelerations and generalized inertia and active forces are substituted into Eq. (91) to solve for generalized constraint forces.

Example 3 is that of the motion of a particle on an elliptical path. The two-dimensional motion of a particle P of mass m in the horizontal plane is shown in Fig. 2. The particle is forced to follow the frictionless elliptic track formed by a rigid wire fixed to an inertial frame $\mathcal{R}$. The coordinate system (x, y) is fixed to $\mathcal{R}$, with

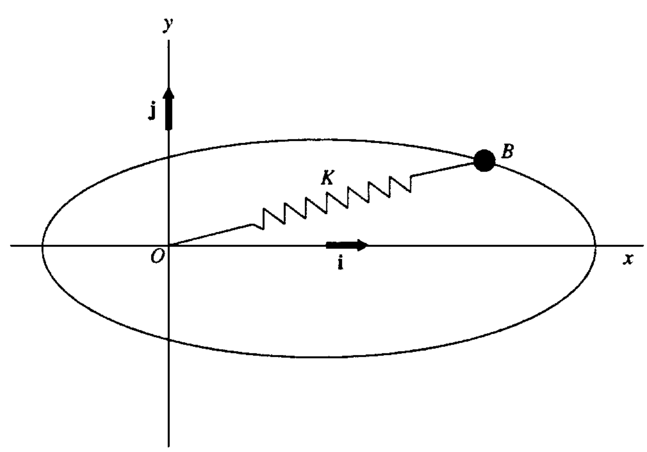

Fig. 2 Schematic for example 3.

its origin at O. A spring with a stiffness constant K and unstretched length of zero is attached from its two ends at the particle P and the point O. We are interested in finding the in-plane constraint force exerted by the wire on the particle P. The elliptic motion of P defines the holonomic constraint

$$a\bar{x}^2 + by^2 = c \tag{97}$$

where $\bar{x} = x - x_0$, such that x_0, a, b, and c are positive constants. Differentiating (97) with respect to t yields

$$a\bar{x}\dot{x} + by\dot{y} = 0 \tag{98}$$

Let the generalized speeds be $u_1 = \dot{x}$ and $u_2 = \dot{y}$. Equation (98) can be written as

$$u_2 = -\epsilon u_1 \tag{99}$$

where $\epsilon = a\bar{x}/by$. Therefore,

$$A = -\epsilon, \qquad B = 0 \tag{100}$$

The matrix A_1 is

$$A_1 = [-A \quad I] = \lfloor \epsilon \quad 1 \rfloor \tag{101}$$

Thus, Eq. (13) for this system is

$$\lfloor \epsilon \quad 1 \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = \frac{abu_1(-yu_1 + \bar{x}u_2)}{(by)^2} \tag{102}$$

The matrix A_2 is

$$A_2 = [I \quad A^T] = \lfloor 1 \quad -\epsilon \rfloor \tag{103}$$

The velocity of P in $\mathcal{R}$ is

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1\boldsymbol{i} + u_2\boldsymbol{j} \tag{104}$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are are unit vectors parallel to the positive x and y axes, respectively. Hence, the partial velocities are $\boldsymbol{v}_1 = \boldsymbol{i}$ and $\boldsymbol{v}_2 = \boldsymbol{j}$. P is subjected to the spring force

$$\boldsymbol{F}_s = -K(x\boldsymbol{i} + y\boldsymbol{j}) \tag{105}$$

Hence, the generalized active forces on P are

$$F_1 = -Kx, \qquad F_2 = -Ky \tag{106}$$

The acceleration of P in $\mathcal{R}$ is

$$^{\mathcal{R}}\boldsymbol{a}^P = \dot{u}_1\boldsymbol{i} + \dot{u}_2\boldsymbol{j} \tag{107}$$

The generalized inertia forces on P are

$$F_1^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_1 = -m\dot{u}_1 \tag{108}$$

$$F_2^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_2 = -m\dot{u}_2 \tag{109}$$

Therefore, the generalized constraint force F^c can be written from Eq. (91) as

$$\begin{Bmatrix} F_1^c \\ F_2^c \end{Bmatrix} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + K \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{110}$$

Equation (15) for this system is

$$\lfloor -m \quad m\epsilon \rfloor \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} = -F_1 + \epsilon F_2 = \left(1 - \frac{a}{b}\right)Kx + \frac{a}{b}Kx_0 \tag{111}$$

Equations (102) and (111) can be used to form the matrix system

$$S\dot{u} = U \tag{112}$$

where

$$S = \begin{bmatrix} \epsilon & 1 \\ -m & m\epsilon \end{bmatrix} \tag{113}$$

and

$$U = \begin{Bmatrix} e_1 \\ e_2 \end{Bmatrix} \tag{114}$$

where

$$e_1 = \frac{abu_1(-yu_1 + \bar{x}u_2)}{(by)^2} \tag{115}$$

$$e_2 = \left(1 - \frac{a}{b}\right)Kx + \frac{a}{b}Kx_0 \tag{116}$$

When $\dot{u}$ is solved for,

$$\dot{u} = S^{-1}U \tag{117}$$

This yields

$$\dot{u}_1 = (1/md)(m\epsilon e_1 - e_2) \tag{118}$$

$$\dot{u}_2 = (1/md)(me_1 + \epsilon e_2) \tag{119}$$

where

$$d = \epsilon^2 + 1 \tag{120}$$

Therefore, the constraint forces are found from Eq. (110) as

$$F_1^c = (1/d)(m\epsilon e_1 - e_2) + Kx \tag{121}$$

$$F_2^c = (1/d)(me_1 + \epsilon e_2) + Ky \tag{122}$$

In vector form

$$\boldsymbol{F}^c = [(1/d)(m\epsilon e_1 - e_2) + Kx]\boldsymbol{i} + [(1/d)(me_1 + \epsilon e_2) + Ky]\boldsymbol{j} \tag{123}$$

For the purpose of comparison, we use the method introduced in Ref. 2 to perform the same task. The system can be described in terms of one generalized coordinate. However, we will consider the same set of generalized coordinates and generalized speeds as earlier, using Eq. (98) to put the system in simple nonholonomic form. Therefore,

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1\boldsymbol{i} + u_2\boldsymbol{j} \tag{124}$$

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1[\boldsymbol{i} - (a\bar{x}/by)\boldsymbol{j}] \tag{125}$$

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1(\boldsymbol{i} - \epsilon\boldsymbol{j}) \tag{126}$$

Hence, the partial velocity is

$$\boldsymbol{v}_1 = \boldsymbol{i} - \epsilon\boldsymbol{j} \tag{127}$$

The acceleration of P is

$$^{\mathcal{R}}\boldsymbol{a}^P = \dot{u}_1(\boldsymbol{i} - \epsilon\boldsymbol{j}) + u_1\frac{ab(\bar{x}u_2 - yu_1)}{(by)^2}\boldsymbol{j} \tag{128}$$

$$^{\mathcal{R}}\boldsymbol{a}^P = \dot{u}_1\boldsymbol{i} + (-\dot{u}_1\epsilon + e_1)\boldsymbol{j} \tag{129}$$

Therefore, the generalized inertia force for the system is

$$F^\star = -m{}^{\mathcal{R}}\boldsymbol{a}^P \cdot \boldsymbol{v}_1 \tag{130}$$

$$F^\star = -m[d\dot{u}_1 - e_1\epsilon] \tag{131}$$

The generalized active force for the system is

$$F = \boldsymbol{F}_s \cdot \boldsymbol{v}_1 = -K(x\boldsymbol{i} + y\boldsymbol{j}) \cdot (\boldsymbol{i} - \epsilon\boldsymbol{j}) \tag{132}$$

$$F = \boldsymbol{F}_s \cdot \boldsymbol{v}_1 = -e_2 \tag{133}$$

Formulating Kane's equation and solving for $\dot{u}_1$ yields

$$\dot{u}_1 = (1/d)e_1\epsilon - (e_2/md) \tag{134}$$

which is identical to Eq. (118). Now, we introduce an auxiliary generalized speed w such that

$$^{\mathcal{R}}\boldsymbol{v}^P = u_1\boldsymbol{v}_1 + w\boldsymbol{\tau} \tag{135}$$

where the partial velocity $\boldsymbol{\tau}$ is a unit vector orthogonal to the elliptic path of P, defined by the holonomic constraint,

$$\boldsymbol{\tau} = (\epsilon\boldsymbol{i} + \boldsymbol{j})\big/\sqrt{d} \tag{136}$$

The constraint force $\boldsymbol{F}^c = F^c\boldsymbol{\tau}$ is found from the equation

$$(-m\,^{\mathcal{R}}\boldsymbol{a}^P + \boldsymbol{F} + \boldsymbol{F}^c)\cdot\boldsymbol{\tau} = 0 \tag{137}$$

which gives

$$-\big(1\big/\sqrt{d}\big)[me_1 + K(\epsilon x + y)] + F^c = 0 \tag{138}$$

Therefore,

$$\boldsymbol{F}^c = \big(1\big/\sqrt{d}\big)[me_1 + K(\epsilon x + y)]\boldsymbol{\tau}$$
$$\boldsymbol{F}^c = (\epsilon/d)[me_1 + K(\epsilon x + y)]\boldsymbol{i} + (1/d)[me_1 + K(\epsilon x + y)]\boldsymbol{j} \tag{139}$$

Figures 3 and 4 show the system response to initial conditions satisfying Eqs. (97) and (99). Also, time simulations for Eqs. (123) and (139) are shown in Fig. 5, and identical values are noticed for constraint forces throughout the trajectory of the system for the same initial conditions. However, the introduction of the auxiliary generalized speed is not needed in the present treatment.

Remark: The values of generalized coordinates and generalized speeds that result from integrating Eqs. (118) and (119) together with the kinematic relations should satisfy the holonomic constraint equation (97) and its kinematic equivalent, Eq. (99), for initial conditions that abide by these equations. However, small violations of these equations are noticed due to the accumulative numerical errors.

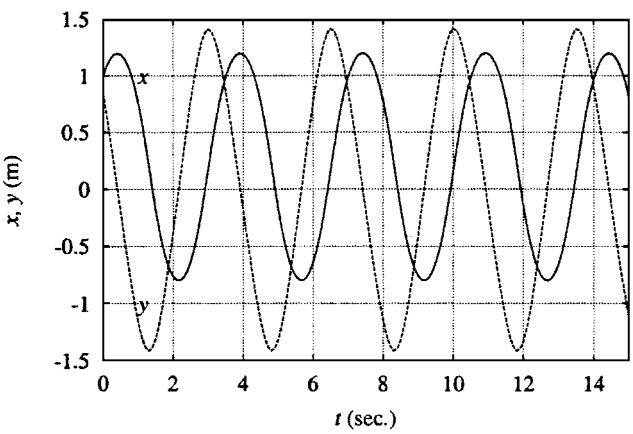

Fig. 3 Example 3: time history of generalized coordinates.

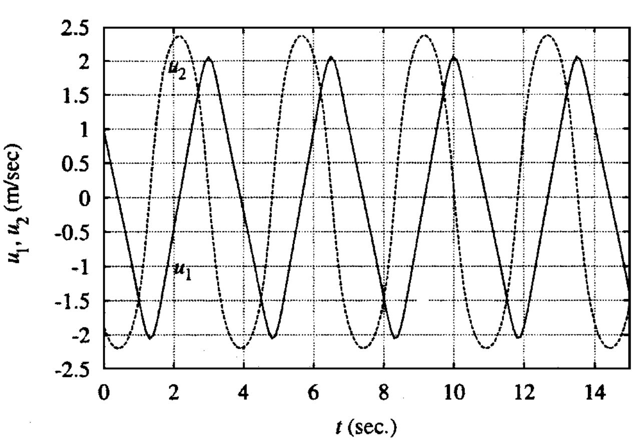

Fig. 4 Example 3: time history of generalized speeds.

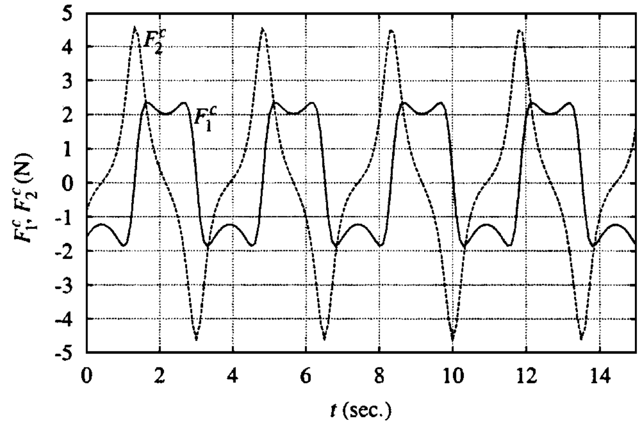

Fig. 5 Example 3: time history of constraint forces.

This problem is common when the equations of motion subjected to constraints are numerically integrated and becomes tangible for long-time simulations. A strategy for overcoming this problem will be presented in a later paper.

Conclusions

When the conformity of the two sets is taken advantage of, the acceleration form of the constraint equations is used with Kane's equations of motion. The resulting set of equations is effective in describing complex constrained motion. This is because of its full-order nature. As a result, it facilitates study of the effects of constraints on the behavior of the dynamic system, by comparing with the corresponding behavior of the same system without constraints. Further advantages include that there is no loss of information, as compared with the reduction of the dimension of the space of generalized speeds. The complexity is decreased relative to an analysis using Lagrange multipliers. If the generalized speeds are chosen such that the constraint matrix is well defined for all configurations, then the constrained inertia matrix inversion involved in the derivation is guaranteed for all values of configurations and velocities that satisfy the constraints. Moreover, this is true without having to assume that the constraint equations are linearly independent, so that any set of constraint equations can be included without the need to extract the largest independent set. The equations appear as first order in generalized speeds and separated in the generalized accelerations. With the kinematic differential equations, they form a complete state-space representation of the system. Finally, explicit expressions for constraint forces are derived without the introduction of auxiliary generalized speeds. This may complement existing approaches based on introduction of auxiliary generalized speeds, that is, fictitious degrees of freedom, or using Lagrange multipliers.

The symbolic manipulator computer program AUTOLEV™ has been adapted to the present method and used to check the validity of the resulting equations of motion obtained in the examples introduced. Simulation results confirm that the models are equivalent. However, it is noticed that enforcing the constraint equations at the acceleration level causes the numerical solutions of the resulting equations of motion to be sensitive to the finite precision and accuracy errors. Thus, it can cause continuous violations of the constraint equations, especially in the case of holonomic constraints because the equations are twice integrated to obtain the generalized coordinates. A remedy to this problem has been formulated and will be presented in a later paper.

References

[1] Kane, T. R., and Wang, C. F., "On the Derivation of Equations of Motion," *Journal of the Society of Industrial and Applied Mathematics*, Vol. 13, No. 2, 1965, pp. 487–492.

[2] Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, 1st ed., Series in Mechanical Engineering, McGraw–Hill New York, 1985.

[3] Mitiguy, P. C., and Kane, T. R., "Motion Variables Leading to Efficient Equations of Motion," *International Journal of Robotics Research*, Vol. 15, No. 5, 1996, pp. 522–532.

[4] Gibbs, J. W., "On the Fundamental Formulas of Dynamics," *American Journal of Mathematics*, Vol. 2, 1879, pp. 49–64.

[5]Appell, P., "Sur les mouvements de roulment; équations du mouvement analougues à celles de Lagrange," *Comptes Rendus*, Vol. 129, 1899, pp. 317–320.

[6]Desloge, E. A., "A Comparison of Kane's Equations of Motion and the Gibbs–Appell Equations of Motion," *American Journal of Physics*, Vol. 54, No. 5, 1986, pp. 470, 471.

[7]Kane, T. R., "Rebuttal to 'A Comparison of Kane's Equations of Motion and the Gibbs–Appell Equations of Motion,'" *American Journal of Physics*, Vol. 54, No. 5, 1986, p. 472.

[8]Desloge, E. A., "Relationship Between Kane's Equations and the Gibbs–Appell Equations," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 1, 1987, pp. 120–122.

[9]Levinson, D. A., "Comment on 'Relationship Between Kane's Equations and the Gibbs–Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, p. 593.

[10]Keat, J. E., "Comment on 'Relationship Between Kane's Equations and the Gibbs–Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 594, 595.

[11]Rosenthal, D. E., "Comment on 'Relationship between Kane's Equations and the Gibbs–Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 595, 596.

[12]Banerjee, A. K., "Comment on 'Relationship Between Kane's Equations and the Gibbs–Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 6, 1987, pp. 596, 597.

[13]Huston, R. L., "Comment on 'Relationship Between Kane's Equations and the Gibbs–Appell Equations,'" *Journal of Guidance, Control, and Dynamics*, Vol. 11, No. 2, 1988, p. 191.

[14]Desloge, E. A., "The Gibbs–Appell Equations of Motion," *American Journal of Physics*, Vol. 56, No. 9, 1988, pp. 841–846.

[15]Desloge, E. A., "Efficacy of the Gibbs–Appell Method for Generating Equations of Motion for Complex Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 114–116.

[16]Sharf, I., D'eleuterio, G. M. T., and Hughes, P. C., "On the Dynamics of Gibbs, Appell, and Kane," *European Journal of Mechanics. A, Solids*, Vol. 11, No. 2, 1992, pp. 145–155.

[17]Huston, R., "On the Equivalence of Kane's Equations and Gibbs' Equations for Multibody Dynamics Formulations," *Mechanical Research Communications*, Vol. 14, No. 2, 1987, pp. 123–131.

[18]Lesser, M., "A Geometrical Interpretation of Kane's Equations," *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, Vol. 436, No. 1896, 1992, pp. 69–87.

[19]Kane, T. R., and Levinson, D. A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control*, Vol. 3, No. 2, 1980, pp. 99–112.

[20]Kane, T. R., and Levinson, D. A., "Use of Kane's Dynamical Equations in Robotics," *International Journal of Robotics Research*, Vol. 2, No. 3, 1983, pp. 3–21.

[21]Kane, T. R., and Levinson, D. A., "Multibody Dynamics," *Journal of Applied Mechanics*, Vol. 50, No. 4b, 1983, pp. 1071–1078.

[22]Rosenthal, D., and Sherman, M., "High Performance Multibody Simulations via Symbolic Equation Manipulation and Kane's Method," *Journal of Astronautical Sciences*, Vol. 34, No. 3, 1986, pp. 223–239.

[23]Rosenthal, D., "An Order *n* Formulation for Robotic Systems," *Journal of Astronautical Sciences*, Vol. 38, No. 4, 1990, pp. 511–529.

[24]Anderson, K., "Recursive Derivation of Explicit Equations of Motion for the Efficient Dynamic/Control Simulation of Large Multibody Systems," Ph.D. Dissertation, Dept. of Mechanical Engineering, Stanford Univ., Stanford, CA, 1990.

[25]Anderson, K., "An Order-N Formulation for the Motion Simulation of General Multi-rigid-bodyConstrained Systems," *Computers and Structures*, Vol. 43, No. 3, 1992, pp. 565–579.

[26]Anderson, K., "An Efficient Formulation for the Modeling of General Multi-flexible-body Constrained Systems," *International Journal of Solids and Structures*, Vol. 30, No. 7, 1993, pp. 921–945.

[27]Anderson, K., "Efficient Modelling of Constrained Multibody Systems for Application with Parallel Computing," *Zeitschrift Fur Angewandte Mathematik und Mechanik*, Vol. 73, No. 6, 1993, pp. 935–939.

[28]Lee, D., Hodges, D., and Patil, M., "Multi-Flexible-Body Dynamic Analysis of Horizontal-Axis Wind Turbines," *Journal of Wind Energy* (to be published).

[29]Pars, L. A., *A Treatise on Analytical Dynamics*, Ox Bow, Woodridge, CT, 1979, pp. 41–44.

[30]Gauss, C. F., "Über ein Neues Allgemeine Grundgesatz der Mechanik," *Journal Für Die Reine Und Angewandte Mathematik*, Vol. 4, 1829, pp. 232–235.

[31]Wehage, R. A., and Haug, E. J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems," *Journal of Mechanical Design*, Vol. 104, Jan. 1982, pp. 247–255.

[32]Shabana, A. A., *Dynamics of Multibody Systems*, 2nd ed., Cambridge Univ. Press, Cambridge, England, U.K., 1998, pp. 251–254.

[33]Wang, J. T., and Huston, R. L., "Kane's Equations With Undetermined Multipiers-Application to Constrained Multibody Systems," *Journal of Applied Mechanics*, Vol. 54, No. 2, 1987, pp. 424–429.

[34]Huston, R., "Constraint Forces and Undetermined Multipliers in Constrained Multibody Systems," *Multibody System Dynamics*, Vol. 3, No. 4, 1999, pp. 381–389.

[35]Su, C.-Y., Oya, M., and Katoh, R., "Robust Adaptive Motion/Force Control of Uncertain Nonholonomic Mechanical Systems," *Proceedings of the American Control Conference*, American Automatic Control Council, Evanston, IL, 2001, pp. 3268–3273.

[36]Gofron, M., and Shabana, A., "Control Structure Interaction in the Nonlinear Analysis of Flexible Mechanical Systems," *Nonlinear Dynamics*, Vol. 4, No. 2, 1993, pp. 183–206.

[37]Gofron, M., and Shabana, A., "Effect of the Deformation in the Inertia Forces on the Inverse Dynamics of Planar Flexible Mechanical Systems," *Nonlinear Dynamics*, Vol. 6, No. 1, 1994, pp. 1–20.

[38]Agrawal, O., and Shabana, A., "Dynamic Analysis of Multibody Systems Using Component Modes," *Computers and Structures*, Vol. 21, No. 6, 1985, pp. 1303–1312.

[39]Yoo, W., and Haug, E., "Dynamics of Flexible Mechanical Systems Using Vibration and Static Correction Modes," *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 108, Sept. 1986, pp. 315–322.

[40]Bauchau, O., and Rodriguez, J., "Modeling of Joints with Clearance in Flexible Multibody Systems," *International Journal of Solids and Structures*, Vol. 39, No. 1, 2002, pp. 41–63.

[41]Udwadia, F. E., and Kalaba, R. E., "A New Perspective on Constrained Motion," *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, Vol. 439, No. 1906, 1992, pp. 407–410.

[42]Udwadia, F. E., and Kalaba, R. E., "On Motion," *Journal of the Franklin Institute*, Vol. 330, No. 3, 1993, pp. 571–577.

[43]Chen, Y. H., "Second-Order Constriants for Equations of Motion of Constrained Systems," *IEEE/ASME Transactions on Mechatronics*, Vol. 3, No. 3, 1998, pp. 240–248.

[44]Udwadia, F. E., and Kalaba, R. E., "The Geometry of Constrained Motion," *Zeitschrift Fur Angewandte Mathematik und Mechanik*, Vol. 75, No. 8, 1995, pp. 637–640.

[45]Udwadia, F. E., "Equations of Motion For Mechanical Systems: A Unified Approach," *International Journal of Non-Linear Mechanics*, Vol. 31, No. 6, 1996, pp. 951–958.

[46]Itiki, C., Kalaba, R. E., and Udwadia, F. E., "Inequality Constraints in the Process of Jumping," *Journal of Applied Mathematics and Computation*, Vol. 78, Nos. 2/3, 1996, pp. 163–173.

[47]Chen, Y. H., "Equations of Motion of Constrained Mechanical Systems: Given Force Depends on Constraint Force," *Mechatronics*, Vol. 9, No. 4, 1999, pp. 411–428.

[48]Chen, Y. H., "Pars's Acceleration Paradox," *Journal of the Franklin Institute*, Vol. 335B, No. 5, 1998, pp. 871–875.